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Variant working vacations on batch arrival queue with reneging and server breakdowns

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Abstract: This paper analyzes a batch arrival infinite-buffer single server variant working vacation queueing system wherein customers arrive according to a Poisson process and the server is subject to breakdown. As soon as the system becomes empty, the server takes working vacation. The service times during regular busy period, working vacation period, vacation times, breakdown times and repair times are assumed to be exponentially distributed and are mutually independent. During working vacations the customer may renege due to impatience. We derive the probability generating function of the steady-state probabilities and obtain the closed form expressions of the system size. In addition, we obtain some other performance measures and monotonicity with respect to K.

Keywords: Queue, Batch arrival, Geometric distribution, Reneging, Variant working vacations, Server breakdowns, Probability generating function

1. Introduction

Queueing systems with server vacation have been investigated extensively due to their wide applications in several areas including computer communication systems, manufacturing and production systems. Vacation models are useful in systems where the server wants to utilize the idle time for different purposes. For more detail on this topic the reader may refer to the surveys of Doshi [6], Takagi [23] and Tian and Zhang [24]. In classical vacation queues, the server completely stops service during the vacation period. However, there are numerous situations where the server remains active during the vacation period which is called working vacation (WV). Servi and Finn [22] introduced this class of semi-vacation policy. They studied an $M/M/1$ queue with multiple working vacations $(MWVs)$. Baba [2] analyzed a $GI/M/1$ queue with MWV. Wu and Takagi [29] generalized Servi and Finn's [22] $M/M/1/WV$ queue to an $M/G/1/WV$ queue. Banik et al. [4] studied a $GI/M/1/N$ WV queue with limited waiting space. Liu et al. [18] derived the stochastic decomposition results in an $M/M/1$ queue with WV.

The bulk input queue models have extensive applications as in computer networks and communication systems the units arrive in batches. For the batch arrival queues, Xu et al. [30] investigated a bulk input $M^X/M/1$ queue with single working vacation. The probability generating function $(p.g.f)$ of the stationary system length distribution is derived using quasi upper triangular transition matrix of two-dimensional Markov chain and matrix analytic method. The stochastic decomposition structure of system length has been derived which indicates the relationship with that of the $M^X/M/1$ queue without vacation. A similar analysis has been carried out in Baba [3] for $M^X/M/1$ queue with MWV and Liu and Song [19] for $M^X/M/1$ queue with working breakdown. A Steady state analysis and computation of the $GI^X/M^b/1/L$ queue with multiple working vacation and partial batch rejection is presented by Yu et al.[33] and Goswami and Vijaya Laxmi [10] analyzed the $GI^X/M/1/N$ queue with single working vacation and partial batch rejection. Recently, a retrial queue with working vacation for the batch arrival $Geo^X/Geo/1$ queue has been analyzed by Upadhyaya [25] by considering the general early arrival system.

The concept of variant multiple vacation policy is relatively a new one where the server is allowed to take a certain fixed number of consecutive vacations, if the system remains empty at the end of a vacation. This kind of vacation schedule is investigated by Zhang and Tian [35] for the $Geo/G/1$ queue with multiple adaptive vacations. Ke

[13] analyzed the operating characteristic of $M^{[X]}/G/1$ system with a variant vacation policy with balking. Banik [5] studied the infinite-buffer single server queue with variant of multiple vacation policy and batch Markovian arrival process by using matrix analytic method. The literature related to this kind of vacation can be identified in papers by Ke and Chang [14], Ke et al. [15] and Wang et al. [28]. In case of WV, Zhang and Hou [36] analyzed a steady state renewal input $GI/M/1/N$ queue with a variant of multiple working vacation by using matrix analytic method. Gao and Yao [7] have developed the variant working vacation (VWV) policy for the $M^X/G/1$ queueing system, where the server operates a randomized vacation policy and takes at most J working vacations when the system becomes empty. A finite buffer $M/M/1$ queue with VWV and balking and reneging has been analyzed by Vijaya Laxmi and Jyothsna [26]. They obtained the steady state probabilities using matrix form solutions.

In real-life, many queueing situations arise where customers tend to be discouraged by a long queue. As a result, customers after joining the queue depart(renege) without getting service. Palm's [20] work seems to be the first to analyze the act of impatient customers in an $M/M/c$ queueing system, where the customers have independent exponential distribution sojourn times. Altman and Yechiali [1] presented the analysis for impatient customers in $M/M/1, M/G/1$ and $M/M/c$ queueing models with server vacations. An $M/M/c$ queue with impatient customers has been studied by Yechiali [32]. Perel and Yechiali [21] considered a 2-phase (fast and slow) Markovian random environment with impatient customers. Yue et al. [34] analyzed the $M/M/1$ queueing system with impatient customers and the variant of multiple vacation policy and obtained the closed-form expressions of the mean system sizes when the server is in different states using probability generating functions. Kim and Kim [17] analyzed a multi-server batch arrival $M^X/M/c$ queue with impatient customers.

Generally, most of the articles in queueing theory deal with systems wherein the servers are reliable, i.e., they do not have the possibility of breakdowns. However, in practice, we often meet the situation where server may breakdown. Phenomena of server breakdowns can be encountered in computer, communication networks, soft manufacturing systems, etc. Grey et al. [[8], [9]] proposed queueing models with service breakdowns. In [8], they analyzed a multiple vacation queueing model, where the service station is subject to breakdown while in operation, whereas, in [9] they considered a general queueing model in which the server may experience several different types of breakdowns. Ke [12] derived some important system characteristics for the N-policy $M/G/1$ queueing system with server vacations, startup, and breakdowns. Wang et al. [27] considered an optimal control of N-policy $M/G/1$ queueing system with server breakdowns and general startup times. Ke and Lin [16] applied the maximum entropy approach to investigate the N-policy $M^{[x]}/G/1$ queueing system with an un-reliable server and a single vacation. Jain and Jain [11] studied WV queue with multiple types of server breakdowns, where each type of breakdown requires a finite random number of stages of repair. Recently, Yang and Wu [31] have investigated an N-policy $M/M/1$ queueing system with WV and server breakdowns.

In real life, there are many situations where customers arrive in groups, e.g., customers arrivals in super markets, restaurants, orders in manufacturing and voice calls in communication centers. In this paper, we consider an $M^X/M/1$ queue with variant working vacations where customers may renege due to impatience and server may breakdown during busy period. On arrival, customers arrive in batches according to a Poisson process and arrival batch size X is a random variable with probability mass function $P(X = l) = b_l$, $l = 1, 2, \cdots$. If there is no customer at the instant of a service completion, the server begins a WV of random length. During the vacation period, the arriving customers are served generally at a lower rate. When a WV ends, the server inspects the system and switches to normal busy period, if there are customers in the queue; otherwise, takes another WV and continues so till K consecutive vacations have been taken. One may note that this VWV generates MWV when $K \to \infty$ and single working vacation (SWV) when K is equal to 1 and after the end of the Kth vacation, the server switches to normal busy period and stays idle or busy depending on the availability of the customers in the system. Further, customers become impatient and renege when the server is on vacation. However, the server may breakdown during the regular busy period and immediately sent for repair which causes interruption in service. If the repair is completed, then the server returns to service immediately. We have obtained the explicit expressions for the steady - state probabilities by using probability generating functions, Various performance measures and the monotonicity on some performance measures with respect to K are discussed.

The rest of the paper is organized as follows. In section 2, model description is given. In Section 3, we have obtained the probability generating functions of the stationary analysis of the system and the closed-form expression of the average system size when the server is in different states are derived. The closed-form expressions of the performance measures,

monotonicity with respect to K are presented in Section 4. Some numerical results are presented in the form of table and graphs in Section 5. Finally, Section 6 concludes the paper

2. Description of the Model

We consider an $M^X/M/1$ queueing system with variant working vacations, reneging and server breakdowns during busy period, wherein customers arrive in batches according to a Poisson process with rate λ. The arrival batch size X is a random variable with probability mass function $P(X = l) = b_l$, $l = 1, 2, \cdots$. The service is provided by a single server with exponential service rate μ.

The server is subject to breakdowns during busy period with a Poisson breakdown rate β. Whenever the server fails, it is immediately repaired at a repair facility, where the repair times have exponential distributions with rate γ. A customer who arrives and finds the server busy or broken down must wait in the queue until the server is available. Although no service occurs during the repair period of a broken server, customers are allowed to join the queue. In case the server breaks down when the service is in progress, he is sent for repair and the customer who has just being served should wait for the server back to complete his remaining service.

At the end of a service, if there is no customer in the system, the server begins a WV of random length which is exponentially distributed with parameter ϕ. During a WV service is provided according to a Poisson distribution with parameter ζ. If the server finds customer at a WV completion instant, it returns to regular busy period; otherwise, the server takes WVs sequentially until 'K' consecutive WVs are complete; after which the server switches to normal busy period staying idle or busy depending on the availability of the customers in the system.

During WV customers become impatient. That is, whenever a batch of customers arrives during WV, an "impatience timer" T is activated, which is exponentially distributed with parameter α. During WV, if the service does not commence before the time T expires, the customer abandons the queue and never returns. Since the arrival and departure of an impatient customer without service are independent, the average reneging rate of a customer is given by $n\alpha$, where n denotes the number of customers in the system. If the server is available in WV before the time T expires, the customer is served with rate ζ. If WV finishes before the impatient timer expires, server switches to normal working period and the customer is served with rate μ.

3. Analysis of the Model

In the next subsection, we develop the difference equations for the probability generating functions ($p.g.f.s$) of the steady state probabilities and solution of the differential equations.

3.1. Steady state Equations

At time t, let $L(t)$ be the number of customers in the system and $J(t)$ denote the status of the server, which is defined as follows:

$$J(t) = \begin{cases} j, \text{ the server is on } (j+1)^{th} \text{ working vacation at time t for } j = 0, 1, \ldots, K-1, \\ K, \text{ the server is idle or busy at time } t, \\ bd, \text{ the server is in breakdown state during busy period at time } t. \end{cases}$$

The process $\{(L(t), J(t)), t \geq 0\}$ defines a continuous-time Markov process with state space

$$\Omega = \{(n, j) : n \geq 0, j = 0, 1, \ldots, K \text{ and } j = bd\}.$$

Let $\pi_{n,j} = \lim_{t \to \infty} \pi\{L(t) = n, J(t) = j\}, n \geq 0, j = 0, 1, \ldots, K \text{ and } j = bd$, denote the steady state probabilities of the process $\{(L(t), J(t)), t \geq 0\}$. Using Markov theory, the set of balance equations are given below:

$$(\lambda + \phi)\pi_{0,0} = (\zeta + \alpha)\pi_{1,0} + \mu\pi_{1,K}, \tag{1}$$

$$(\lambda + \phi + \zeta + \alpha)\pi_{1,0} = \lambda b_1 \pi_{0,0} + (\zeta + 2\alpha)\pi_{2,0}, \tag{2}$$

$$(\lambda + \phi + \zeta + n\alpha)\pi_{n,0} = \lambda \sum_{m=1}^{n} b_m \pi_{n-m,0} + (\zeta + (n+1)\alpha)\pi_{n+1,0}, \ \ n \geq 2, \tag{3}$$

$$(\lambda + \phi)\pi_{0,j} = (\zeta + \alpha)\pi_{1,j} + \phi\pi_{0,j-1}, 1 \leq j \leq K-1, \tag{4}$$

$$(\lambda + \phi + \zeta + \alpha)\pi_{1,j} = \lambda b_1 \pi_{0,j} + (\zeta + 2\alpha)\pi_{2,j}, 1 \leq j \leq K-1, \tag{5}$$

$$(\lambda + \phi + \zeta + n\alpha)\pi_{n,j} = \lambda \sum_{m=1}^{n} b_m \pi_{n-m,j} + (\zeta + (n+1)\alpha)\pi_{n+1,j}, 1 \leq j \leq K-1, n \geq 2, \tag{6}$$

$$\lambda \pi_{0,K} = \phi\pi_{0,K-1}, \tag{7}$$

$$(\lambda + \mu + \beta)\pi_{1,K} = \lambda b_1 \pi_{0,K} + \mu\pi_{2,K} + \phi\sum_{j=0}^{K-1}\pi_{1,j} + \gamma\pi_{1,bd}, \tag{8}$$

$$(\lambda + \mu + \beta)\pi_{n,K} = \lambda \sum_{m=1}^{n} b_m \pi_{n-m,K} + \mu\pi_{n+1,K} + \phi\sum_{j=0}^{K-1}\pi_{n,j} + \gamma\pi_{n,bd}, \ n \geq 2, \tag{9}$$

$$(\lambda + \gamma)\pi_{1,bd} = \beta\pi_{1,K}, \tag{10}$$

$$(\lambda + \gamma)\pi_{n,bd} = \lambda \sum_{m=1}^{n-1} b_m \pi_{n-m,bd} + \beta\pi_{n,K}, \ n \geq 2, \tag{11}$$

and the normalizing condition is

$$\sum_{n=0}^{\infty}\sum_{j=0}^{K}\pi_{n,j} + \sum_{n=1}^{\infty}\pi_{n,bd} = 1. \tag{12}$$

The state probabilities are obtained by solving the equations (1) to (11) using $(p.g.f.s)$. Let us define the $p.g.f$ of $\pi_{n,j}$ as

$$G_j(z) = \sum_{n=0}^{\infty}\pi_{n,j}z^n, 0 \leq z \leq 1, j = 0,1,...,K.$$

$$G_{bd}(z) = \sum_{n=1}^{\infty}\pi_{n,bd}z^n.$$

Define $G'_j(z) = \dfrac{d}{dz}G_j(z) = \sum_{n=1}^{\infty}nz^{n-1}\pi_{n,j}, j = 0,1,...,K$ and $j = bd$, and $p.g.f$ of the arrival batch size X is

$G(z) = \sum_{l=1}^{\infty}b_l z^l$, $|z| \leq 1$ with $G(1) = \sum_{l=1}^{\infty}b_l = 1$.

We assume that the arrival batch size X follows a geometric distribution with parameter q, that is, $P(X=l) = (1-q)^{l-1}q, 0 < q < 1 (\ l = 1,2,\cdots)$. It is easy to observe that

$$G(z) = \frac{qz}{1-(1-q)z}. \tag{13}$$

Now, multiplying equations (1), (2) and (3) by z^n, and summing over all possible values of n and re-arranging the terms, we get

$$\alpha z(1-z)G'_0(z) + (\lambda z(G(z)-1) - (\phi+\zeta)z + \zeta)G_0(z) = \zeta(1-z)\pi_{0,0} - \mu z\pi_{1,K}. \tag{14}$$

Similarly, from equations (4), (5) and (6) and (7), (8) and (9), respectively, we get

$$\alpha z(1-z)G'_j(z) + (\lambda z(G(z)-1) - (\phi+\zeta)z + \zeta)G_j(z) = \zeta(1-z)\pi_{0,j} - \phi z\pi_{0,j-1}, \ 1 \leq j \leq K-1. \tag{15}$$

$$(\lambda z(G(z)-1)+(1-z)\mu - \beta z)G_K(z)+\phi z\sum_{j=0}^{K-1}G_j(z) = \quad (\mu(1-z)-\beta z)\pi_{0,K}$$

$$+ z\left[\mu\pi_{1,K}+\phi\sum_{j=0}^{K-2}\pi_{0,j}\right]-\gamma z G_{bd}(z). \tag{16}$$

And from equations (10) and (11), we obtain

$$\left[\lambda(1-G(z))+\gamma\right]G_{bd}(z) = \beta\left[G_K(z)-\pi_{0,K}\right] \tag{17}$$

By taking $z=1$ in equations (14) and (15), we obtain

$$\phi G_0(1) = \mu\pi_{1,K} \tag{18}$$

and

$$G_j(1) = \pi_{0,j-1}.1 \le j \le K-1, \tag{19}$$

which implies that

$$\phi\sum_{j=0}^{K-1}G_j(1) = \mu\pi_{1,K}+\phi\sum_{j=1}^{K-1}\pi_{0,j-1}. \tag{20}$$

Equation (14) can be written as

$$G_0'(z)+\left[\frac{\lambda(G(z)-1)}{\alpha(1-z)}-\frac{\phi+\zeta}{\alpha(1-z)}+\frac{\zeta}{\alpha z(1-z)}\right]G_0(z) = \frac{\zeta}{\alpha z}\pi_{0,0}-\frac{\mu}{\alpha(1-z)}\pi_{1,K}. \tag{21}$$

The above equation is a linear differential equation whose solution is given by

$$G_0(z) = \frac{(1-(1-q)z)^{\frac{-\lambda}{\alpha(1-q)}}}{(1-z)^{\frac{\phi}{\alpha}}z^{\frac{\zeta}{\alpha}}}\left[\frac{\zeta}{\alpha}I_1(z)\pi_{0,0}-\frac{\mu}{\alpha}I_2(z)\pi_{1,K}\right], \tag{22}$$

where

$$\left.\begin{array}{l}I_1(z) = \int_0^z(1-(1-q)x)^{\frac{\lambda}{\alpha(1-q)}}(1-x)^{\frac{\phi}{\alpha}}x^{\frac{\zeta}{\alpha}-1}dx,\\[3mm]I_2(z) = \int_0^z(1-(1-q)x)^{\frac{\lambda}{\alpha(1-q)}}(1-x)^{\frac{\phi}{\alpha}-1}x^{\frac{\zeta}{\alpha}}dx.\end{array}\right\} \tag{23}$$

Proceeding similarly, equation (15) gives

$$G_j(z) = \frac{(1-(1-q)z)^{\frac{-\lambda}{\alpha(1-q)}}}{(1-z)^{\frac{\phi}{\alpha}}z^{\frac{\zeta}{\alpha}}}\left[\frac{\zeta}{\alpha}I_1(z)\pi_{0,j}-\frac{\phi}{\alpha}I_2(z)\pi_{0,j-1}\right],1 \le j \le K-1. \tag{24}$$

Our aim is to get $\pi_{1,K}$ and $\pi_{0,j}$ in terms of $\pi_{0,0}$. We observe that $z=1$ and $z=0$ are the roots of the denominator of the right hand side of equations (22) and (24), we have $z=1$ and $z=0$ must be the roots of the numerator of the right hand side of those equations. Therefore

$$\pi_{1,K} = \frac{\zeta}{\mu}\frac{I_1(1)}{I_2(1)}\pi_{0,0} \tag{25}$$

and

$$\pi_{0,j} = \frac{\phi}{\zeta}\frac{I_2(1)}{I_1(1)}\pi_{0,j-1},\ 1 \le j \le K-1. \tag{26}$$

Equation (26) can be written as

$$\pi_{0,j} = C^j\pi_{0,0},1 \le j \le K-1, \tag{27}$$

where $C = \frac{\phi}{\zeta}\frac{I_2(1)}{I_1(1)}$. Using equations (7) and (27), we obtain

$$\pi_{0,K} = \frac{\phi}{\lambda} C^{K-1} \pi_{0,0}. \tag{28}$$

Using equations (25) and (27) in equations (22) and (24) respectively, we get

$$G_0(z) = \frac{(1-(1-q)z)^{\frac{-\lambda}{\alpha(1-q)}}}{(1-z)^{\frac{\phi}{\alpha}} z^{\frac{\zeta}{\alpha}}} \left[I_1(z) - I_2(z) \frac{I_1(1)}{I_2(1)} \right] \frac{\zeta}{\alpha} \pi_{0,0}, \tag{29}$$

$$G_j(z) = \frac{(1-(1-q)z)^{\frac{-\lambda}{\alpha(1-q)}}}{(1-z)^{\frac{\phi}{\alpha}} z^{\frac{\zeta}{\alpha}}} \left[\frac{I_2(1)}{I_1(1)} I_1(z) - I_2(z) \right] \frac{\phi}{\alpha} C^{j-1} \pi_{0,0}, 1 \le j \le K-1. \tag{30}$$

Next, we derive the probabilities $\pi_{0,0}$ and $\pi_{0,K}$. Let $G_j(1) = \sum_{n=0}^{\infty} \pi_{n,j}$, for $j = 0,1,\ldots,K$. From equations (29) and (30), using L'Hospital rule, we get

$$G_j(1) = C^{j-1} \pi_{0,0}, j = 0,1,\ldots,K-1. \tag{31}$$

Using equations (17) and (20), we obtain

$$G_K(z) = \frac{\phi z \left[\lambda(1-G(z)) + \gamma \right] \sum_{j=0}^{K-1} \left[G_j(z) - G_j(1) \right] - N_1 \pi_{0,K}}{D_1(z)}, \tag{32}$$

where

$$N_1 = \left(\mu(1-z) - \beta z \right) \lambda(1-G(z)) + \mu(1-z)\gamma$$

and

$$D_1(z) = \lambda z(1-G(z)) \left[\lambda(1-G(z)) + \beta + \gamma \right] - \mu(1-z) \left[\lambda(1-G(z)) + \gamma \right].$$

Taking $z = 1$ and applying L'Hospital rule, we get

$$G_K(1) = \frac{\phi\gamma \sum_{j=0}^{K-1} G'_j(1) + (\mu\gamma - \lambda\beta G'(1))\pi_{0,K}}{(\mu\gamma - \lambda(\beta+\gamma)G'(1))}. \tag{33}$$

Since the average number of customers in the system during WVs is $E[L_j] = \sum_{j=0}^{K-1} G'_j(1)$. We first derive $E[L_j]$ for $j = 0,1,\ldots,K-1$. From equations (14) and (18), we have for $z = 1$ and using L'Hospital rule

$$(\lambda+\phi)G'_0(1) = (\lambda G'(1) - \zeta)G_0(1) + \zeta\pi_{0,0}. \tag{34}$$

Similarly, from equations (15) and (19), we get

$$(\lambda+\phi)G'_j(1) = (\lambda G'(1) - \zeta)G_j(1) + \zeta\pi_{0,j}, j = 1,2,\ldots,K-1. \tag{35}$$

Equations (34) and (35) imply

$$E[L_j] = G'_j(1) = \frac{(\lambda G'(1) - \zeta)}{\alpha+\phi} G_j(1) + \frac{\zeta}{\alpha+\phi} \pi_{0,j}, j = 0,1,\ldots,K-1. \tag{36}$$

Using equations (31) and (27), equation (36) can be written as

$$E[L_j] = G'_j(1) = \frac{(\lambda G'(1) - \zeta)C^{j-1} + \zeta C^j}{\alpha+\phi} \pi_{0,0}, j = 0,1,\ldots,K-1. \tag{37}$$

Therefore, the mean system size when the server is on WV, denoted by $E[L_{WV}]$, is obtained as

$$E[L_{WV}] = \sum_{j=0}^{K-1} E[L_j] = \left[\frac{\lambda G'(1) - \zeta(1-C)}{\alpha+\phi} \right] \frac{(1-C^K)}{C(1-C)} \pi_{0,0}. \tag{38}$$

Substituting equation (38) and (28) in equation (33), we get

$$G_K(1) = \left[\frac{\phi\gamma(\lambda G'(1) - \zeta(1-C))}{(\mu\gamma - \lambda(\beta+\gamma)G'(1))(\alpha+\phi)} \frac{(1-C^K)}{C(1-C)} + \frac{(\mu\gamma - \lambda\beta G'(1))}{(\mu\gamma - \lambda(\beta+\gamma)G'(1))} \frac{\phi}{\lambda} C^{K-1} \right] \pi_{0,0}, \quad (39)$$

and then from equation (17), we get

$$G_{bd}(1) = (\beta/\gamma)(G_K(1) - \pi_{0,K}). \quad (40)$$

The normalizing condition (12) can be written as

$$\sum_{j=0}^{K-1} G_j(1) + G_K(1) + G_{bd}(1) = 1. \quad (41)$$

Substituting equations (31), (39) and (40) in (41), we get

$$\pi_{0,0} = \frac{\lambda(\mu\gamma - \lambda(\beta+\gamma)G'(1))(\alpha+\phi)h(K)}{\lambda D_2 + \phi\mu\gamma(\alpha+\phi)H(K)} \quad (42)$$

where

$$D_2 = \alpha(\mu\gamma - \lambda(\beta+\gamma)G'(1)) + \phi[\mu\gamma - \zeta(1-C)(\beta+\gamma)],$$

$$h(K) = (C(1-C))/(1-C^K)$$

and

$$H(K) = (C^K(1-C))/(1-C^K).$$

Substituting equation (42) in equations (28) and (38), we obtain the probability of an idle server and the mean system size when the server is on WV as follows:

$$\pi_{0,K} = \frac{\phi(\mu\gamma - \lambda(\beta+\gamma)G'(1))(\alpha+\phi)H(K)}{\lambda D_2 + \phi\mu\gamma(\alpha+\phi)H(K)} \quad (43)$$

and

$$E[L_{WV}] = \frac{\lambda(\mu\gamma - \lambda(\beta+\gamma)G'(1))(\lambda G'(1) - \zeta(1-C))}{\lambda D_2 + \phi\mu\gamma(\alpha+\phi)H(K)}. \quad (44)$$

Note that $E[L_K]$ represents the mean system size when the server is busy or idle and $E[L_{bd}]$ represents the mean system size when the server is breakdown and under repair, which can be found from the following theorem.

Theorem 3.1 *The mean system sizes during busy / idle and breakdown periods can be expressed as*

$$E[L_K] = \frac{\phi}{(\mu\gamma - \lambda(\beta+\gamma)G'(1))} \left(A_1 E[L_{WV}] + A_2 \pi_{0,0} + \frac{A_3}{\phi} \pi_{0,K} \right),$$

and

$$E[L_{bd}] = G'_{bd}(1) = \frac{\beta}{\gamma} \left[E[L_K] + \frac{\lambda G'(1)(\phi E[L_{WV}] + \lambda G'(1)\pi_{0,K})}{(\mu\gamma - \lambda G'(1)(\beta+\gamma))} \right]$$

where

$$A_1 = \left(\frac{\mu\gamma^2 + \beta(\lambda G'(1))^2 + \lambda\gamma(\beta+\gamma)(G''(1)/2)}{(\mu\gamma - \lambda(\beta+\gamma)G'(1))} + \frac{\gamma(\lambda G'(1) - (\phi+\zeta+\alpha))}{(\phi+2\alpha)} \right)$$

$$A_2 = \frac{\lambda\gamma(2G'(1) + G''(1))}{2(\phi+2\alpha)h(K)} \text{ and } A_3 = \frac{\lambda\mu\gamma^2(2G'(1) + G''(1)) + 2\beta(\lambda G'(1))^3}{2(\mu\gamma - \lambda G'(1)(\beta+\gamma))}$$

Proof. From Eq. (32), using L'Hospital rule, we obtain

$$E[L_K] = G_K'(1) = \frac{1}{2(\mu\gamma - \lambda G'(1)(\beta+\gamma))^2}\left[\phi\gamma(\mu\gamma - \lambda G'(1)(\beta+\gamma))\sum_{j=0}^{K-1}G_j''(1)\right]$$
$$+ \frac{\phi\gamma\lambda G''(1)(\beta+\gamma) + 2\phi(\mu\gamma^2 + \beta(\lambda G'(1))^2)}{2(\mu\gamma - \lambda G'(1)(\beta+\gamma))^2}\sum_{j=0}^{K-1}G_j'(1) \qquad (45)$$
$$+ \frac{\lambda\mu\gamma^2(2G'(1) + G''(1)) + 2\beta(\lambda G'(1))^3}{2(\mu\gamma - \lambda G'(1)(\beta+\gamma))^2}\pi_{0,K},$$

where $G_j''(1)$ is obtained by differentiating $G_j(z)$ twice at $z=1$ for $j=0,1,\ldots,K-1$. Differentiating twice (13) and (14) and taking $z=1$, we get

$$G_j''(1) = (2(\lambda G'(1) - (\phi+\zeta+\alpha))G_j'(1) + \lambda(2G'(1) + G''(1))G_j(1))/(\phi+2\alpha). \qquad (46)$$

Substituting Eq. (46) in (45), we obtain

$$E[L_K] = \phi\left(\frac{\mu\gamma^2 + \beta(\lambda G'(1))^2 + \lambda\gamma(\beta+\gamma)(G''(1)/2)}{(\mu\gamma - \lambda(\beta+\gamma)G'(1))^2} + \frac{\gamma(\lambda G'(1) - (\phi+\zeta+\alpha))}{(\mu\gamma - \lambda G'(1)(\beta+\gamma))(\phi+2\alpha)}\right)E[L_{WV}]$$
$$+ \frac{\phi\gamma}{2(\mu\gamma - \lambda G'(1)(\beta+\gamma))}\frac{\lambda(2G'(1) + G''(1))}{(\phi+2\alpha)}\frac{1-C^K}{C(1-C)}\pi_{0,0} \qquad (47)$$
$$+ \frac{\lambda\mu\gamma^2(2G'(1) + G''(1)) + 2\beta(\lambda G'(1))^3}{2(\mu\gamma - \lambda G'(1)(\beta+\gamma))^2}\pi_{0,K},$$

where $\pi_{0,0}$, $\pi_{0,K}$ and $E[L_{WV}]$ are calculated by using Eqs. (42), (43) and (44). Differentiating Eq. (17) with respect to z, taking $z=1$ and using (33), we get the mean system size when the server is breakdown and under repair $E[L_{bd}]$ as,

$$E[L_{bd}] = G_{bd}'(1) = \frac{\beta}{\gamma}\left[E[L_K] + \frac{\lambda G'(1)(\phi E[L_{WV}] + \lambda G'(1)\pi_{0,K})}{(\mu\gamma - \lambda G'(1)(\beta+\gamma))}\right]. \qquad (48)$$

$\square$

From equation (13), we obtain that $G'(1) = g = 1/q$, $G''(1) = 2(1-q)/q^2$ and $2G'(1) + G''(1) = 2/q^2$ and $\rho = (\lambda g)/\mu$. Therefore, equations (44), (47) and (48) can be written as

$$E[L_{WV}] = \frac{\lambda\mu(\gamma - \rho(\beta+\gamma))(\mu\rho - \zeta(1-C))}{\lambda\alpha\mu(\gamma - \rho(\beta+\gamma)) + \phi[\lambda(\mu\gamma - \zeta(1-C)(\beta+\gamma)) + \mu\gamma(\alpha+\phi)H(K)]}, \qquad (49)$$

$$E[L_K] = \phi\left(\frac{\gamma(\mu\rho - (\phi+\zeta+\alpha))}{\mu(\gamma - \rho(\beta+\gamma))(\phi+2\alpha)} + \frac{\mu(q\gamma)^2 + \beta(q\mu\rho)^2 + \lambda\gamma(\beta+\gamma)(1-q)}{(\mu q(\gamma - \rho(\beta+\gamma)))^2}\right)E[L_{WV}]$$
$$+ \frac{1}{\mu q^2(\gamma - \rho(\beta+\gamma))}\left[\frac{\phi\lambda\gamma}{(\phi+2\alpha)}\frac{1-C^K}{C(1-C)}\pi_{0,0} + \frac{\lambda\mu\gamma^2 + \beta q^2(\mu\rho)^3}{(\gamma - \rho(\beta+\gamma))}\pi_{0,K}\right], \qquad (50)$$

and

$$E[L_{bd}] = \frac{\beta}{\gamma}\left[E[L_K] + \frac{\mu\rho(\phi E[L_{WV}] + \mu\rho\pi_{0,K})}{\mu(\gamma - \rho(\beta+\gamma))}\right]. \qquad (51)$$

Where the probabilities $\pi_{0,0}$ and $\pi_{0,K}$ are calculated by using equations (42) and (43) as follows:

$$\pi_{0,0} = \frac{\lambda\mu(\gamma - \rho(\beta+\gamma))(\alpha+\phi)h(K)}{\lambda\alpha\mu(\gamma - \rho(\beta+\gamma)) + \phi[\lambda(\mu\gamma - \zeta(1-C)(\beta+\gamma)) + \mu\gamma(\alpha+\phi)H(K)]}, \qquad (52)$$

$$\pi_{0,K} = \frac{\phi\mu(\gamma - \rho(\beta+\gamma))(\alpha+\phi)H(K)}{\lambda\alpha\mu(\gamma - \rho(\beta+\gamma)) + \phi[\lambda(\mu\gamma - \zeta(1-C)(\beta+\gamma)) + \mu\gamma(\alpha+\phi)H(K)]}. \qquad (53)$$

Let L be the number of customers in the system. The mean system size $E[L] = E[L_{WV}] + E[L_K] + E[L_{bd}]$ can be calculated from equations (49), (50) and (51).

4. Performance Measures

In this section, some other performance measures and their monotonicity with respect to K are presented.

• When the system is in state (n, j), $n \geq 0$, $j = 0, 1, \cdots, K-1$, the rate of abandonment of a customer due to impatience is $n\alpha$. Thus, the average rate of abandonment due to impatience (R_a) is given by

$$R_a = \sum_{j=0}^{K-1} \sum_{n=1}^{\infty} n\alpha\pi_{n,j} = \alpha E[L_{WV}]. \tag{54}$$

• From equation (53), the probability that the server is idle is given by

$$\pi_{0,K} = \frac{\phi\mu(\gamma - \rho(\beta + \gamma))(\alpha + \phi)H(K)}{\lambda\alpha\mu(\gamma - \rho(\beta + \gamma)) + \phi[\lambda(\mu\gamma - \zeta(1-C)(\beta + \gamma)) + \mu\gamma(\alpha + \phi)H(K)]}$$

If we consider K as a continuous variable and take the derivative of $\pi_{0,K}$ with respect to K,

we have $\dfrac{d\pi_{0,K}}{dK} < 0$ and the inequality follows from the fact that $H(K)$ decreases with K.

Therefore, $\pi_{0,K}$ is a decreasing function of K.

• Let π_{WV} be the probability when the server is on WV. From equation (31), we have

$$\pi_{WV} = \sum_{j=0}^{K-1} G_j(1) = \frac{1 - C^K}{C(1-C)}\pi_{0,0}. \tag{55}$$

Substituting equation (42) in (55), we get

$$\pi_{WV} = \frac{\lambda\mu(\gamma - \rho(\beta + \gamma))(\alpha + \phi)}{\lambda\alpha\mu(\gamma - \rho(\beta + \gamma)) + \phi[\lambda(\mu\gamma - \zeta(1-C)(\beta + \gamma)) + \mu\gamma(\alpha + \phi)H(K)]}. \tag{56}$$

We see that π_{WV} increases with K because of the decrease of $\pi_{0,K}$ with respect to K.

• The probability of busy server is given by

$$\pi_b = \sum_{n=1}^{\infty} \pi_{n,K} = G_K(1) - \pi_{0,K}. \tag{57}$$

Substituting equations (31), (49) and (53) in the above equation, we obtain

$$\pi_b = \frac{\phi\lambda\gamma[(\alpha + \phi)(1/q)H(K) + \mu\rho - \zeta(1-C)]}{\lambda\alpha\mu(\gamma - \rho(\beta + \gamma)) + \phi[\lambda(\mu\gamma - \zeta(1-C)(\beta + \gamma)) + \mu\gamma(\alpha + \phi)H(K)]}. \tag{58}$$

From the fact that $\pi_{0,K}$ decreases, π_{WV} increases and $\dfrac{d\pi_b}{dK} < 0$ with K, we find that π_b

decreases with K.

• The probability when the server is breakdown and under repair are given respectively, by

$$\pi_{bd} = \sum_{n=1}^{\infty} \pi_{n,bd} = \frac{\beta}{\gamma}(G_K(1) - \pi_{0,K}), \tag{59}$$

$$\pi_{bd} = \frac{\phi\lambda\beta[\mu\rho - \zeta(1-C) + (\alpha + \phi)(1/q)H(K)]}{\lambda\alpha\mu(\gamma - \rho(\beta + \gamma)) + \phi[\lambda(\mu\gamma - \zeta(1-C)(\beta + \gamma)) + \mu\gamma(\alpha + \phi)H(K)]}. \tag{60}$$

• The probability when the system is empty and the server is on WV is given by

$$\pi_e = \sum_{j=0}^{K-1} \pi_{0,j} = \frac{(1 - C^K)}{(1-C)}\pi_{0,0}. \tag{61}$$

Using equation (55) in (61), we get

$$\pi_e = C\pi_{WV}.$$ (62)

• The average number of customers in the queue $E[L_q]$ is given by

$$E[L_q] = \sum_{j=0}^{K}\sum_{n=1}^{\infty}(n-1)\pi_{n,j} + \sum_{n=1}^{\infty}n\pi_{n,bd}\,,,$$ (63)

and it can be written as

$$E[L_q] = E[L] - \left[1 - \left(\pi_e + \pi_{0,K} + \pi_{bd}\right)\right].$$ (64)

5. Numerical Analysis

In this section, to study the parameter impact on the system performance, numerical computations are carried out and a few of those are presented in the form of tables and graphs. For example, a flour mill, where the machine grinds the grain like rice, wheat, pulses etc., into flour. Carbs/orders arrive in batches according to a Poisson process with rate λ. The arrival batch size X is a random variable with probability mass function $P(X = l) = b_l$, $l = 1, 2, \cdots$, Flour times follows an exponential distribution with rate μ. When the service completed, if there are no orders in queue the flour machine performs preventive maintenance with exponential distribution with rate ϕ, during this maintenance period the flour machine still works at a lower rate ζ. When the maintenance ends, if there are orders in the queue, the machine will come into normal flour rate μ, otherwise; it takes another maintenance task and continuous a finite consecutive maintenance tasks. Maintenance tasks during maintenance are lubrication like oil and grease etc., into machinery parts.

During maintenance period the orders may renege due to impatience. Whenever orders arrive during maintenance period an impatience timer is activated with exponential distribution. If the service does not commence before an impatience timer, then the orders renege from the queue with rate $n\alpha$ and never returns, where n denotes number of orders in the queue. If flour machine is available during maintenance period before the timer expires, the orders will be served with rate ζ. If maintenance period ends before an impatience times expires, then machine changes to normal service rate. A break down of the flour machine may happen at any time according to Poisson distribution with rate β during busy period. Whenever the flour machine undergoes breakdown, it is sent for repair immediately. The repair times follow exponential distribution with rate γ. During repair period, the flour machine will not work until repair is completed and orders still arrive according to Poisson process. Once repair is completed, the flour machine becomes active and resumes its service.

We consider the parameters as $\lambda = 1.8$, $\mu = 6.0$, $K = 5$, $\alpha = 0.5$, $\zeta = 1.2$, $\phi = 3.0$, $\beta = 0.8$, $\gamma = 8.5$ and $q = 0.7$ for all the figures and tables, unless they are considered as variables or their values are mentioned in the respective figures and tables. Table 1 gives the effect of variant working vacation K on different performance measures. The performance measures $\pi_{0,K}$, π_{bd} and π_b are decreasing and π_{WV}, π_e, $E[L_{wv}]$ and R_a are increasing function of K, as it should be.

Table 1: Effect of K on different performance measures

K	$\pi_{0,K}$	π_{WV}	π_e	π_{bd}	π_b	$E[L_{wv}]$	R_a
1	0.3002505	0.2643400	0.1801503	0.0374545	0.3979548	0.1653439	0.0826719
2	0.1833275	0.3982300	0.2713977	0.0359950	0.3824473	0.2490918	0.1245459
3	0.1166647	0.4745665	0.3234218	0.0351629	0.3736058	0.2968400	0.1484200
4	0.0760744	0.5210470	0.3550987	0.0346562	0.3682223	0.3259135	0.1629567
5	0.0503631	0.5504893	0.3751639	0.0343352	0.3648122	0.3443295	0.1721647
6	0.0336669	0.5696083	0.3881937	0.0341268	0.3625978	0.3562884	0.1781442
7	0.0226493	0.5822247	0.3967919	0.0339893	0.3611365	0.3641800	0.1820900

The impact of arrival rate on $E[L]$ is shown in " Figure 1", for different mean batch sizes $(g = 1/q)$. It can be observed that as λ increases, $E[L]$ increases for fixed q. For fixed λ, $E[L]$ increases with g as it should be. " Figure 2" depicts the impact of probabilities during regular busy period (π_b, π_{bd}) and WV (π_{WV}) versus λ for different q. We observe that π_b and π_{bd} increase as λ increases, whereas π_{WV} slightly increases and then decreases. For fixed λ, as q increases, π_b and π_{bd} decrease whereas π_{WV} increases. The intersect points shows that π_b and π_{WV} are equal at a particular λ corresponding to the q.
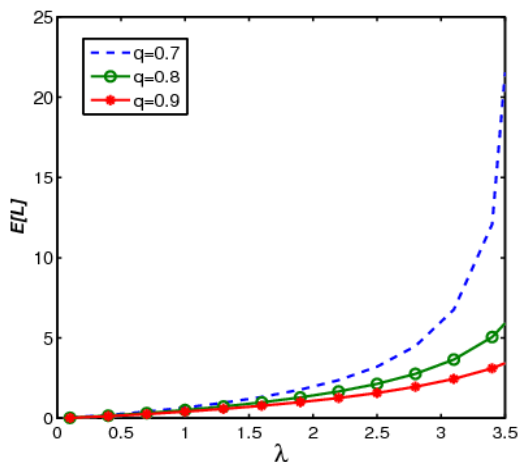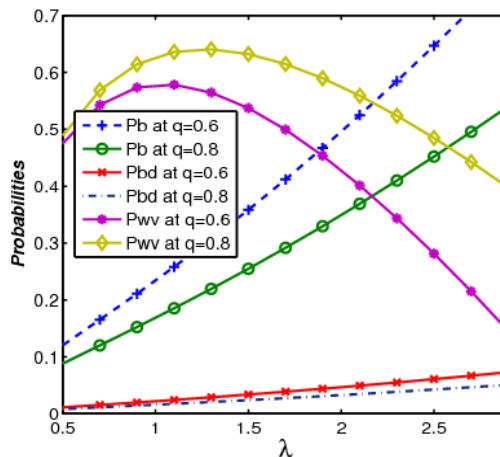

Figu. 1. Effect of λ on $E[L]$

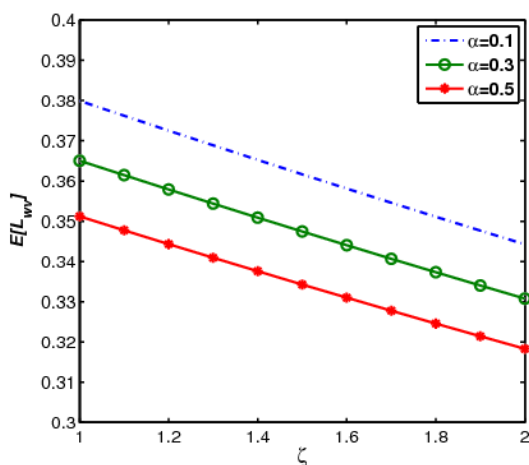**Fig. 2. Effect of λ on π_b and π_{WV}**

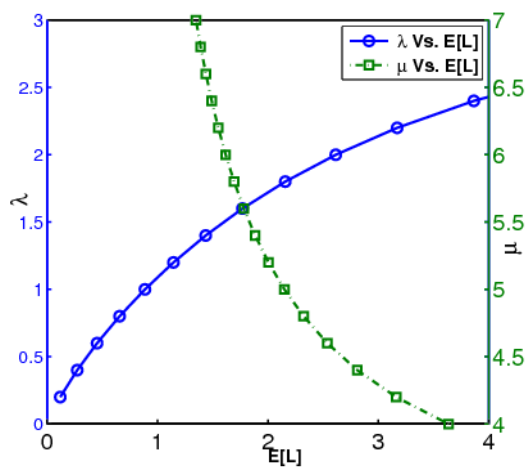**Fig. 3. Effect of ζ on $E[L_{wv}]$**

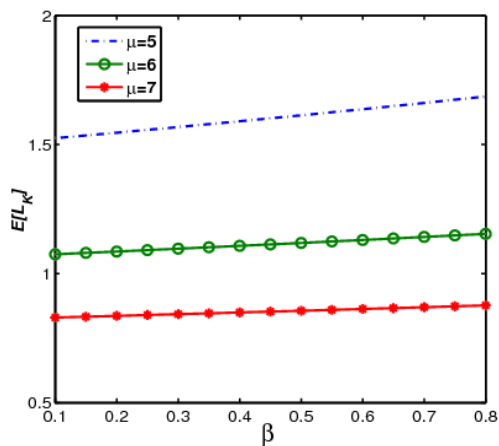**Fig. 4. Effect of λ and μ on $E[L]$**
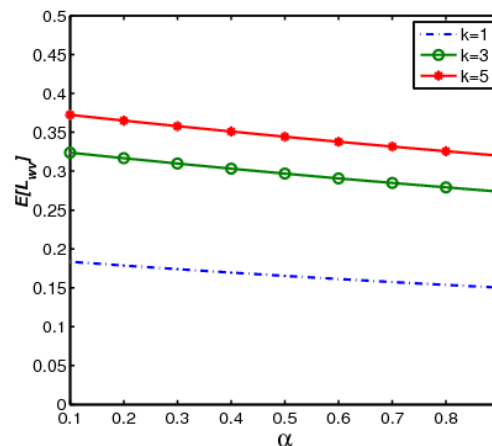
Fig. 5. Effect of β on $E[L_K]$

Fig. 6. Effect of α on $E[L_{WV}]$

"Figure 3" plots the impact of service rate during a WV period ζ on $E[L_{WV}]$ for variant reneging rates (α). We observe that $E[L_{WV}]$ decreases with increase of both ζ and α. The effect of λ and μ on $E[L]$ is depicted in " Figure 4". As expected, one may observe that $E[L]$ increases as λ increases whereas it decreases with μ. Further, the intersection point of the two curves at $(\lambda,\mu)=(1.5,5.5)$ gives us the optimum value of $E[L]$. " Figure 5" shows the effect of β on $E[L_K]$ for different μ values. As expected, one may observe that $E[L_K]$ increases as β increases whereas it decreases with μ. " Figure 6" shows the impact of reneging rate (α) on $E[L_{WV}]$ for different K values. It can be observed that as α increases $E[L_{WV}]$ decreases. We can also observe that single WV has better performance than MWV for the above parameters. Table 2 shows that the effect of breakdown and reneging on mean system size $E[L]$. Observe that $E[L]$ decreases as α increases and $E[L]$ increases as β increases.

Table 2: The effect of β and α on $E[L]$

$\beta \backslash \alpha$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	1.614433	1.568982	1.527160	1.488487	1.452572	1.419093	1.387781	1.358409	1.330784
0.3	1.661921	1.615710	1.573183	1.533852	1.497320	1.463261	1.431402	1.401512	1.373395
0.5	1.711287	1.664312	1.621074	1.581079	1.543925	1.509278	1.476866	1.401512	1.373395
0.7	1.762639	1.714893	1.670939	1.630275	1.592491	1.557253	1.524279	1.493331	1.464211
0.9	1.816091	1.767571	1.722894	1.681554	1.643135	1.607297	1.573755	1.542270	1.512636

6. Conclusions

In this paper, we have studied an $M^X/M/1$ queueing system with variant WVs, reneging and server breakdowns during busy period. We have derived probability generating functions of the number of customers in the system and the corresponding mean system sizes when the server is in different states. We have derived closed-form expressions for some other performance measures, the rate of abandonment due to impatience. The effect of some parameters on the performance measures of the system have been investigated and the results are presented in the form of tables and graphs. The effects of reneging and breakdown paramers on the performance of the model have been shown. The technique adopted in this paper can be applied to analyze models like $M/M^X/1$ queue with variant working vacations and server breakdowns, inpatient customer $M^X/M/c$ queue with variant working vacation and un-reliable servers, etc.

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